On the Laplacian and Signless Laplacian Spectra of Complete Multipartite Graphs

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Article Info

Article history:

Received Jul 12th, 2017
Revised Aug 20th, 2017
Accepted Oct 26th, 2017

Keyword:

Laplacian spectra
Signless Laplacian spectra
Polynomial characteristics
Bipartite graph
Complete Multipartite graph

ABSTRACT

Let G be a finite simple graph with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G is an $(n \times n)$-matrix $A(G) = [a_{ij}]$ where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ elsewhere, and the degree matrix of G is a diagonal $(n \times n)$-matrix $D(G) = [d_{ij}]$ where $d_{ii} = \deg_G(v_i)$ and $d_{ij} = 0$ for $i \neq j$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. The study of spectrum of Laplacian and signless Laplacian matrix of graph are interesting topic till today. In this paper, we determine the Laplacian and signless Laplacian spectra of complete multipartite graphs.

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1. INTRODUCTION

Let G be a finite graph without loop and multiple edges of order n ($n \geq 1$). Let its vertex set is $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set is $E(G)$. The adjacency matrix of G is an $(n \times n)$-matrix $A(G) = [a_{ij}]$ where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ elsewhere [1]. The degree matrix of G is a diagonal $(n \times n)$-matrix $D(G) = [d_{ij}]$ where $d_{ii} = \deg_G(v_i)$ and $d_{ij} = 0$ for $i \neq j$ [2]. The Laplacian matrix $L(G)$ of G is defined by $L(G) = D(G) - A(G)$ and the signless Laplacian matrix $Q(G)$ of G is defined by $Q(G) = D(G) + A(G)$ [3].

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ where $\lambda_1 > \lambda_2 > \ldots > \lambda_k$ are distinct eigenvalues from a matrix of graph G and let $m(\lambda_1)$, $m(\lambda_2)$, …, $m(\lambda_k)$ are algebraic multiplicities of λ_i, $i = 1, 2, \ldots, k$. The spectrum of graph G is $(2 \times k)$-matrix that contains $\lambda_1, \lambda_2, \ldots, \lambda_k$ for the first row and $m(\lambda_1), m(\lambda_2), \ldots, m(\lambda_k)$ for the second row and denoted by $Spec(G)$ [3]. The Laplacian spectrum of G is the spectrum of Laplacian matrix of G and denoted by $Spec_L(G)$. The signless Laplacian spectrum of G is the spectrum of signless Laplacian matrix of graph G and denoted by $Spec_Q(G)$.

Some results on Laplacian and signless Laplacian spectrum of graphs have been reported, for examples the properties of Laplacian spectra of a graph and Laplacian integral graphs [4, 5], the Laplacian spectrum of complex networks [6], the Laplacian spectrum of graph obtained from K_l by adhering the root of isomorphic trees T to every vertex of K_l [7], the Laplacian spectrum of weakly quasi-threshold graphs [8], the signless Laplacian spectrum of coronae [9], the Laplacian spectrum of non-commuting of dihedral group [10], the Laplacian spectrum of some graphs [11], the Laplacian spectra of graphs and complex networks [12], the (signless) Laplacian spectral of the line graphs of lollipop graphs [13], the Laplacian spectrum of neural networks [14], and the Laplacian spectra of product graphs [15]. Several studies relating to Laplacian and signless Laplacian matrix have been reported such as [16-23]. In this paper, we determine the Laplacian and signless Laplacian spectra of complete multipartite graphs.

Graph G is said to be complete n-partite if vertex set of G can be partitioned into n partite sets $V_1, V_2, \ldots, V_n$ such that if $u \in V_i$ and $v \in V_j$ then $uv \in E(G)$ for $i \neq j$. A complete n-partite graph for some integer $n \geq 2$ is called a complete multipartite graph[1]. If $|V_i| = \alpha_i$ for all i ($i = 1, 2, \ldots, n$) then graph G be denoted by $K(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$. If $\alpha_i = t$ for all i then the complete n-partite graph $K(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ be denoted by $K_n(t)$.

So, the complete n-partite graph $K_n(1)$ is the complete graph K_n. If $\alpha_i = t$ for i = 1, 2, …, n-1 and $\alpha_n = s$ then we denoted $K_{n-1}(t)(s)$ instead of $K(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$.

2. RESEARCH METHOD

We use the library research method to determine Laplacian and signless Laplacian spectra of complete multipartite graphs. We begin by observing specific cases of Laplacian and signless Laplacian spectra of complete multipartite graphs. The next step is analyzing the polynomial and spectrum pattern of these specific cases to obtain general pattern. The last is present the result as a theorem with its proof.

3. RESULTS AND DISCUSSION

Now, we present our results on the Laplacian spectra for several complete multipartite graphs.

Theorem 1: The Laplacian spectrum of complete bipartite graph $K_2(m)$ for $m \geq 2$ and $m \in \mathbb{N}$ is

$$spec_L\big(K_2(m)\big) = \begin{bmatrix} 2m & m & 0 \\ 1 & 2(m-1) & 1 \end{bmatrix}$$

Proof: By eliminating matrix ($L(K_2(m))$ - λI) to an upper triangular matrix, we have the characteristic polynomial of Laplacian matrix $L(K_2(m))$ is $p(\lambda) = \lambda(\lambda - m)^{2(m-1)}(\lambda - 2m)$. Therefore, eigenvalues of $L(K_2(m))$ are $2m$, m and 0, and their algebraic multiplicities are 1, 2(m-1) and 1, respectively. So, we have the complete proof. □

Theorem 2: The Laplacian spectrum of complete tripartite graph $K_3(m)$ for $m \geq 2$ and $m \in \mathbb{N}$ is

$$spec_L\big(K_3(m)\big) = \begin{bmatrix} 3m & 2m & 0 \\ 2 & 3(m-1) & 1 \end{bmatrix}$$

Proof: The characteristic polynomial of Laplacian matrix $L(K_2(m))$ is $p(\lambda) = \lambda(\lambda - 2m)^{3(m-1)}(\lambda - 3m)^2$. □

For more general case, we have the following theorem.

Theorem 3: The Laplacian spectrum of complete multipartite graph $K_n(m)$ for $m \geq 2$ and $m \in \mathbb{N}$ is

$$spec_L\big(K_n(m)\big) = \begin{bmatrix} nm & (n-1)m & 0 \\ n-1 & n(m-1) & 1 \end{bmatrix}$$

Proof: It is enough to show the characteristic polynomial of $L(K_n(m))$. The degree of all vertices in complete multipartite $K_n(m)$ is $(n-1)m$. So, the degree matrix of $K_n(m)$ is $D(K_n(m)) = (n-1)mI$, where I is identity matrix of order nm. According to the definition of Laplacian matrix, we have $L(K_n(m)) = D\big(K_n(m)\big) - A(K_n(m))$. Appling Gauss-Jordan elimination method for $L\big(K_n(m)\big) - \lambda I$, we will have a diagonal matrix with no zero in its main diagonal. By multiplying all elements in the main diagonal, we have the characteristic polynomial of $L(K_n(m))$ is $p(\lambda) = \lambda(\lambda - (n-1)m)^{n(m-1)}(\lambda - nm)^{n-1}$. It completes the proof. □

Theorem 4: The Laplacian spectrum of complete bipartite graph $K(m, m+1)$ for $m \geq 2$ and $m \in \mathbb{N}$ is

$$spec_L\big(K(m, m+1)\big) = \begin{bmatrix} 2m+1 & m+1 & m & 0 \\ 1 & m-1 & m & 1 \end{bmatrix}$$

Proof: We can compute that characteristic polynomial of Laplacian matrix $L(K(m, m+1))$ is $p(\lambda) = \lambda(\lambda - m)^m\big(\lambda - (m+1)\big)^{m-1}(\lambda - (2m+1))$. □

Theorem 5: The Laplacian spectrum of complete multipartite graph $K_n(1)(2)$ for $n \in \mathbb{N}$ is

$$spec_L\big(K_n(m)\big) = \begin{bmatrix} n+2 & n & 0 \\ n & 1 & 1 \end{bmatrix}$$

Proof: It is easy to check that characteristic polynomial of Laplacian matrix $L(K_n(1)(2))$ is $p(\lambda) = \lambda(\lambda - n)(\lambda - (n+2))^n$. □

The following two theorems are special case for the next theorem.

Theorem 6: The Laplacian spectrum of complete multipartite graph $K_n(2)(3)$ for $n \in \mathbb{N}$ is

$$spec_L(K_n(2)(3)) = \begin{bmatrix} 2n+3 & 2n+1 & 2n & 0 \\ n & n & 2 & 1 \end{bmatrix}$$

Proof: The characteristic polynomial of Laplacian matrix $L(K_n(2)(3))$ is $p(\lambda) = \lambda(\lambda - 2n)^2(\lambda - (2n+1))^n(\lambda - (2n+3))^n$. □

Theorem 7: The Laplacian spectrum of complete multipartite graph $K_n(3)(4)$ for $n \in \mathbb{N}$ is

$$spec_L(K_n(3)(4)) = \begin{bmatrix} 3n+4 & 3n+1 & 3n & 0 \\ n & 2n & 3 & 1 \end{bmatrix}$$

Proof: The characteristic polynomial of Laplacian matrix $L(K_n(3)(4))$ is $p(\lambda) = \lambda(\lambda - 3n)^3(\lambda - (3n+1))^{2n}(\lambda - (3n+4))^n$. □

From Theorem 6 and Theorem 7, if we set $n = 1$ then we have $K_1(2)(3) = K(2)(3)$ and $K_1(3)(4) = K(3)(4)$. We see that the Laplacian spectra for these two graphs can be computed using Theorem 4. For more general result, we have the following theorem.

Theorem 8: The Laplacian spectrum of complete multipartite graph $K_n(m)(m+1)$ for $m, n \in \mathbb{N}$ and $m \geq 2$ is

$$spec_L(K_n(m)(m+1)) = \begin{bmatrix} nm+m+1 & nm+1 & nm & 0 \\ n & n(m-1) & m & 1 \end{bmatrix}$$

Proof: We can observe that characteristic polynomial of Laplacian matrix $L(K_n(1)(2))$ is

$$p(\lambda) = \lambda(\lambda - mn)^m(\lambda - (mn+1))^{(m-1)n}(\lambda - (mn+m+1))^n$$

The following theorem can be used to determine the Laplacian spectrum of complete multipartite graph $K(1, 2, 3, \ldots, n)$ for $n \in \mathbb{N}$ and $n \geq 2$.

Theorem 9: The characteristic polynomial of $L(K(1, 2, 3, \ldots, n)$ for $n \in \mathbb{N}$ and $n \geq 2$ is

$$p(\lambda) = \lambda\left(\lambda - \left(\frac{n(n+1)}{2} - n\right)\right)^{n-1}\left(\lambda - \left(\frac{n(n+1)}{2} - (n-1)\right)\right)^{n-2}\cdots\left(\lambda - \left(\frac{n(n+1)}{2}\right)\right)\left(\lambda - \left(\frac{n(n+1)}{2}\right)\right)^{n-1}$$

Proof: Using Gaussian elimination method to Laplacian matrix $L(K(1, 2, 3, \ldots, n)$ will leads to the desired result. □

Theorem 10: The signless Laplacian spectrum of complete bipartite graph $K_2(m)$ for $m \in \mathbb{N}$ and $m \geq 2$ is

$$spec_L(K_2(m)) = \begin{bmatrix} 2m & m & 0 \\ 1 & 2(m-1) & 1 \end{bmatrix}$$

Proof: By the fact that Laplacian spectrum and signless Laplacian spectrum of bipartite graphs are always equal, then the proof follows from Theorem 1. □

Theorem 11: The signless Laplacian spectrum of complete tripartite graph $K_3(m)$ for $m \in \mathbb{N}$ and $m \geq 2$ is

$$spec_L(K_2(m)) = \begin{bmatrix} 4m & 2m & m \\ 1 & 3(m-1) & 2 \end{bmatrix}$$

Proof: We can observe that characteristic polynomial of signless Laplacian matrix $Q(K_3(m))$ is
$p(\lambda) = (\lambda - 4m)(\lambda - 2m)^{3(m-1)}(\lambda - m)^2$. □

Theorem 12: The signless Laplacian spectrum of complete bipartite graph $K(m, m+1)$ for $m \geq 2$ and $m \in \mathbb{N}$ is

$$spec_L(K(m, m+1)) = \begin{bmatrix} 2m+1 & m+1 & m & 0 \\ 1 & m-1 & m & 1 \end{bmatrix}$$

Proof: By the fact that Laplacian spectrum and signless Laplacian spectrum of bipartite graphs are always equal, then the proof follows from Theorem 4. □

The following theorem is presented without proof. It is easy to be observed from the characteristic polynomial of the Laplacian matrix of $K_n(1)(2)$ for $n \in \mathbb{N}$.

Theorem 13: The Laplacian spectrum of complete multipartite graph $K_n(1)(2)$ for $n \in \mathbb{N}$ is

$$spec_L(K_n(m)) = \begin{bmatrix} \frac{3n}{2} + \sqrt{\frac{13n-6}{4}} & n & \frac{3n}{2} - \sqrt{\frac{13n-6}{4}} \\ 1 & n & 1 \end{bmatrix}$$

CONCLUSION

We have determined the Laplacian and signless Laplacian spectra of several complete multipartite graphs. Because there are many kinds of complete multipartite graphs according to the cardinality of each partition sets, so the further studies can be done to determine the Laplacian and signless Laplacian spectra of another complete multipartite graphs. Further studies also can be done to determine another spectrum of these graphs.

ACKNOWLEDGMENTS

We sincerely thank the Faculty of Science and Technology UIN Maulana Malik Ibrahim Malang for funding the research.

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