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Some sequence spaces on the Orlicz space

Haryadi[1], Supama[2], Atok Zulijanto[3]
[1] Departement of Mathematics, Universitas Muhammadiyah Palangkaraya
[2,3] Departement of Mathematics, Universitas Gadjah Mada

Email: haryadi_ump@yahoo.co.id, supama@ugm.ac.id, atokzulijanto@ugm.ac.id

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ABSTRACT

Although, the Orlicz space is a generalization of classical Banach space L_p, the method used in [4] to construct sequences spaces on L_p no longer work on the Orlicz space. In this paper, we construct some sequence space in the Orlicz space such that the resulting sequences spaces is a generalization of the sequence space in L_p. We use modular to construct the Luxemburg norm and describe some topological properties in the spaces.

Corresponding Author:

Third Author,
Departement of Mathematics,
UIN Maulana Malik Ibrahim Malang,
Jl. Gajayana No. 50 Malang, Jawa Timur, Indonesia 65144
Email: xxxxxxx@gmail.com

1. INTRODUCTION

It is well known that the Orlicz space is a generalization of the classical Banach space L_p. Although we can use the method in [4] to construct sequences space in L_p, this method is no longer work in generalized Orlicz space L_ϕ, for, if for every sequence (u_k) of element of L_ϕ we define the function $p_\phi\big((u_k)\big) = \sup\limits_k \phi^{-1}\left(\int_E \phi\big(u_k(x)\big)dx\right)$, then p_ϕ in general is not a norm, since it is may not homogenous.

The construction of scalar-valued sequences space using Orlicz function can be found in [1], [5] and [7]. In [8], Yilmas constructed vector-valued sequence space in modular space. The recent result on Cesaro sequence space in L_ϕ for the Orlicz function which satisfies Δ_2-condition can be found in [2].

In this paper, fist we construct some sequence space on the Orlicz space using modular ρ_ϕ and its Luxemburg norm. Secondly, we describe some topological properties on these spaces. Now, we introduce some preliminaries.

As usual, the set of all positive integers, the real number system, and the set of all non-negative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$, respectively. Given a linear space X over the real field $\mathbb{R}$. The function $\rho: X \to [0,\infty]$ is called modular if it satisfies (i) $\rho(x) = 0$ if and only if $x = 0$, (ii) $\rho(-x) = \rho(x)$ for each $x \in X$ and (iii) $\rho(ax + by) \le \rho(x) + \rho(y)$ for each $x, y \in X$ and for each $a, b \ge 0$, $a + b = 1$. Furthermore, if $\rho(ax + by) \le a\rho(x) + a\rho(y)$, then ρ is said convex modular.

An Orlicz function is a function $\phi: \mathbb{R} \to [0,\infty)$ which satisfies (i) ϕ is even, (ii) $\phi(x) = 0$ if and only if $x = 0$, (ii) ϕ continuous and convex. An Orlicz function ϕ is said to satisfy the Δ_2-condition if there is a

$K > 0$ such that $\phi(2x) \leq K\phi(x)$ for each $x \geq 0$. (see e.g., [3]). Note that the Orlicz function is monotonic increasing on $\mathbb{R}^+$. If $\lambda > 1$, the the Δ_2-condition implies there exist $K(\lambda) > 0$ such that $\phi(\lambda x) \leq K(\lambda)\phi(x)$ for each $x \geq 0$.

Let S be the set of all Lebesgue measurable real-valued function on $E \subset \mathbb{R}$. The generalized Orlicz space L_ϕ is defined as

$$L_\phi = \left\{u \in S: \int_E \phi(\lambda u(x))dx, for\ some\ \lambda > 0\right\}.$$

It is easy to show that the function

$$\rho_\phi(u) = \int_E \phi(u(x))dx$$

is a convex modular on L_ϕ. Furthermore, the space L_ϕ is a Banach space with respect to the Luxemburg norm

$$\|u\|_\phi = \inf\left\{t > 0: \rho_\phi\left(\frac{u}{t}\right) \leq 1\right\}.$$

In addition, it is true that (i) $\|u\|_\phi \leq 1$ implies $\rho_\phi(u) \leq \|u\|_\phi$ and (ii) $\|u\|_\phi > 1$ implies $\rho_\phi(u) \geq \|u\|_\phi$ (see, e.g., [3], [6]).

A sequence (u_k) in L_ϕ is called convergent (norm-convergent) to $u_0 \in L_\phi$ if $\|u_k - u_0\|_\phi \to 0$ as $k \to \infty$; it is called ρ_ϕ-convergent (modular-convergent) to $u_0 \in L_\phi$ if there exists $\lambda > 0$ such that $\rho_\phi(\lambda(u_k - u_0)) \to 0$ as $k \to \infty$. The condition $\|u_k - u_0\|_\phi \to 0$ as $k \to \infty$ is equivalent to the condition $\rho_\phi(\lambda(u_k - u_0)) \to 0$ if $k \to \infty$ for every $\lambda > 0$. Hence, the norm-convergence implies the ρ_ϕ-convergent. However, the ρ_ϕ-convergent does not always imply the norm-convergence. Furthermore, if the Orlicz function satisfy the Δ_2-condition, then the norm-convergence is equivalent to ρ_ϕ-convergent (see, e.g.,[3]).

2. RESEARCH METHOD

In this research, we develop the existing mathematical theoretical results using literature study. Instead of norm, we use modular to construct the sequences spaces. Motivated by the construction of the sequence spaces on Banach space, we extend the definition the sequence spaces on the Orlicz space L_ϕ. Furthermore, to find some topological properties, we use Luxemburg norm defined by modular on the the sequence spaces.

3. RESULTS

Let $\omega(L_\phi)$ be the space of all sequence in L_ϕ. The member of $\omega(L_\phi)$ is written as (u_k) and the sequence of zero function is denoted by (θ_k). A sequence of element of $\omega(L_\phi)$ is written as (u^n), i.e. $({u^n}_k) \in \omega(L_\phi)$ for each n. Now we introduce our main sequence spaces. For any Orlicz function ϕ, we define the following sequence spaces:

$$c_0(L_\phi) = \left\{(u_k): \lim_{k\to\infty} \rho_\phi(\lambda u_k) = 0 \text{ for some } \lambda > 0\right\}$$

$$c(L_\phi) = \{(u_k): (u_k - u_0) \in c_0(L_\phi) \text{ for some } u_0 \in L_\phi\}$$

$$l_\infty(L_\phi) = \left\{(u_k): \sup_k \rho_\phi(\lambda u_k) < \infty\ for\ some\ \lambda > 0\right\}.$$

Using the fact that ρ_ϕ is a modular, it is easy to show that each $c_0(L_\phi)$, $c(L_\phi)$ and $l_\infty(L_\phi)$ is a linear space.

Theorem 1. If the Orlicz function ϕ satisfies the Δ_2-condition then

(i) $c_0(L_\phi) = \left\{(u_k): \lim_{k\to\infty} \rho_\phi(u_k) = 0\right\}$

(ii) $c(L_\phi) = \{(u_k): (u_k - u_0) \in c_0(L_\phi) \text{ for some } u_0 \in L_\phi\}$

(iii) $l_\infty(L_\phi) = \left\{(u_k): \sup_k \rho_\phi(u_k) < \infty\right\}$.

Proof. We prove (iii). The proof of (i) and (ii) are similar. Let $l_\infty^0(L_\phi)$ denotes the right side of (iii). It is obvious that $l_\infty^0(L_\phi) \subseteq l_\infty(L_\phi)$. Let (u_k) be any elemen of $l_\infty(L_\phi)$. There exist $\lambda > 0$ such that $\sup_k \rho_\phi(\lambda u_k) < \infty$. If $\lambda \geq 1$, the monotonicity of ρ_ϕ implies $\sup_k \rho_\phi(u_k) \leq \sup_k \rho_\phi(\lambda u_k) < \infty$, i.e. $(u_k) \in l_\infty^0(L_\phi)$. If $\lambda < 1$, since ϕ satisfying the Δ_2-condition, then there exit $K(\lambda) > 0$ such that $\phi(\frac{1}{\lambda}x) \leq K(\lambda)\phi(x)$ for each $x \geq 0$. Consequently, $\sup_k \rho_\phi(u_k) \leq K(\lambda).\sup_k \rho_\phi(\lambda u_k) < \infty$, which mean that $(u_k) \in l_\infty^0(L_\phi)$. ■

Note that for the Orlicz function $\phi(x) = |x|^p$, $1 < p < \infty$, the quantity $\rho_\phi(u_k)$ is equal to $\int_E |u_k(x)|^p dx$, so the sequence spaces we have constructed above are a generalization of the corresponding sequence spaces in L_p.

Theorem 2. The function $\rho_\infty\colon l_\infty(L_\phi) \to [0, \infty)$ where $\rho_\infty((u_k)) = \sup_k \rho_\phi(u_k)$ is a convex modular.

Proof. It is clear that $\rho_\infty((u_k)) = 0$ if and only if $(u_k) = (\theta_k)$ and $\rho_\infty((-u_k)) = \rho_\infty((u_k))$.
Let $a, b \geq 0$, $a + b = 1$ and $(u_k), (v_k) \in l_\infty(L_\phi)$. Since ρ_ϕ is modular, then $\rho_\phi(au_k + bv_k) \leq \rho_\phi(u_k) + \rho_\phi(v_k)$ for each $k \in \mathbb{N}$, and implies $\rho_\infty(a(u_k) + b(v_k)) \leq \rho_\infty((u_k)) + \rho_\infty((v_k))$. Hence ρ_∞ is a modular. Furthermore, by convecity of ρ_ϕ, we have $\rho_\phi(au_k + bv_k) \leq a\rho_\phi(u_k) + b\rho_\phi(v_k)$ for each $k \in \mathbb{N}$, so we have $\rho_\infty(a(u_k) + b(v_k)) \leq a\rho_\infty((u_k)) + b\rho_\infty((v_k))$. ∎

Since ρ_∞ is a convex modular, we can defined Luxemburg norm $\|.\|_{\rho_\infty}$ on $l_\infty(L_\phi)$ as follows:
$$\|(u_k)\|_{\rho_\infty} = inf\left\{t > 0 \colon \rho_\infty\left(\frac{(u_k)}{t}\right) \leq 1\right\}.$$

Given $(u_k), (v_k) \in \omega(L_\phi)$. We write $|(u_k)| \leq |(v_k)|$ if for each k, $|u_k| \leq |v_k|$ almost everywhere on E. A set $W \subset \omega(L_\phi)$ is called solid, if $(v_k) \in W$ and $|(u_k)| \leq |(v_k)|$ implies $(u_k) \in W$.

Theorem 3. For any Orlicz function ϕ, each of $c_0(L_\phi)$ and $l_\infty(L_\phi)$ is solid in $\omega(L_\phi)$.

Proof. Given $(u_k), (v_k)$ such that $|(u_k)| \leq |(v_k)|$. By monotonicity of ϕ, we have $\rho_\phi(\lambda u_k) \leq \rho_\phi(\lambda v_k)$ for every $\lambda > 0$. If $(v_k) \in c_0(L_\phi)$, then $\lim_{k\to\infty} \rho_\phi(\lambda u_k) \leq \lim_{k\to\infty} \rho_\phi(\lambda v_k) = 0$, hence $(u_k) \in c_0(L_\phi)$. If $(v_k) \in l_\infty(L_\phi)$, then $\sup_k \rho_\phi(\lambda u_k) \leq \sup_k \rho_\phi(\lambda v_k) < \infty$, and implies $(u_k) \in l_\infty(L_\phi)$. ∎

In other hand, $c(L_\phi)$ is not solid in $\omega(L_\phi)$. For example, let $\phi(x) = |x|^p$, $p \geq 1$ and let (u_k) be a sequence of real-valued function on $[a, b]$, where $u_k(x) = 1$ if k is even and $u_k(x) = 0$ if k is odd. Define the sequence (v_k) where $v_k(x) = 1$, $x \in [a, b]$ for each k. Then $(v_k) \in c(L_\phi)$ and $|(u_k)| \leq |(v_k)|$, but $(u_k) \notin c(L_\phi)$.

Theorem 4. For any Orlicz function ϕ, the space $(l_\infty(L_\phi), \|.\|_{\rho_\infty})$ is a complete.

Proof. Let (u^n) be a Cauchy sequence in $l_\infty(L_\phi)$ and given any $\lambda > 0$. Note that for each n, $u^n = (u_k^n)$ is in $l_\infty(L_\phi)$. Then $\rho_\infty(\lambda(u^m - u^n)) \to 0$ as $n, m \to \infty$. Since $\rho_\phi(\lambda((u_k^m - u_k^n)) \leq \rho_\infty(\lambda(u^m - u^n))$ for each $k \in \mathbb{N}$, then $\rho_\phi(\lambda((u_k^m - u_k^n)) \to 0$ as $n, m \to \infty$, i.e. $(u_k^n)_n$ is Cauchy sequence in L_ϕ. The completeness of L_ϕ implies there exist $u_k^0 \in L_\phi$ such that $u_k^n \to u_k^0$ for each $k \in \mathbb{N}$. Let we define the sequence (u_k^0). Given any $\epsilon > 0$. Fix m such that $\rho_\phi(2\lambda(u_k^m - u_k^n)) < \epsilon/2$ and take n_0 such that $\rho_\phi(2\lambda(u_k^0 - u_k^m)) < \epsilon/2$. For each k, we have
$$\rho_\phi(\lambda(u_k^0 - u_k^n)) \leq \tfrac{1}{2}\rho_\phi(2\lambda(u_k^0 - u_k^m)) + \tfrac{1}{2}\rho_\phi(2\lambda(u_k^m - u_k^n)) < \tfrac{1}{2}\epsilon$$
if $n \geq n_0$. Consequently, for each $n \geq n_0$ we have $\rho_\infty(\lambda(u^0 - u^n)) < \epsilon$. Since this is true for any $\lambda > 0$, this mean that (u^n) convergence to (u_k^0). Furthermore, since $\rho_\phi(\tfrac{1}{2}u_k^0) \leq \rho_\phi(u_k^0 - u_k^n) + \rho_\phi(u_k^n) < \epsilon + M$ for some $M < \infty$, then we have $\rho_\infty(\tfrac{1}{2}u_k^0) < \infty$, and so $(u_k^0) \in l_\infty(L_\phi)$. ∎

Is the case the Orlicz function satisfies the Δ_2-condition, we have the following result.

Theorem 5. If the Orlicz function ϕ satisfies the Δ_2-condition, the norm-convergence and modular-convergence is equivalent.

Proof. We only need to proof that modular-convergence implies norm-convergence. Suppose that (u^n) $\rho_\infty - convergent$, namely to (u_k). Given any $\lambda > 0$. Since ϕ satisfies the Δ_2-condition, there exists an integer r

such that $\phi(\lambda x) \leq K^r \phi(x))$ for each $x \geq 0$. Let $\epsilon > 0$ be arbitrary. Take a natural number n_0 such that for each $n \geq n_0$, $\rho_\infty(u^n - (u_k)) < \frac{\epsilon}{2K^r}$. Then for each k we have $\rho_\phi(u_k^n - u_k) < \frac{\epsilon}{2K^r}$, and therefore

$$\rho_\phi\big(\lambda(u_k^n - u_k)\big) \leq K^r \rho_\phi(u_k^n - u_k) < \epsilon/2$$

for each $n \geq n_0$. Hence, $\rho_\infty(\lambda(u^n - (u_k)) < \epsilon$ for each $n \geq n_0$, i.e. (u^n) norm-convergence. ∎

Theorem 6. For any Orlicz function ϕ, $\mathrm{c}(\mathrm{L}_\phi)$ is a closed subspace of $\mathrm{l}_\infty(\mathrm{L}_\phi)$.

Proof. Let (u^n) be a sequence in $\mathrm{c}(\mathrm{L}_\phi)$ such that $u^n \to (u_k)$ as $n \to \infty$. Then $\rho_\infty(\lambda(u^n - (u_k))) \to 0$ for any $\lambda > 0$. Hence, $\rho_\phi(\lambda(u_k^n - u_k)) \to 0$ as $n \to \infty$. Since for each n, $(u_k^n) \in \mathrm{c}(\mathrm{L}_\phi)$, then there exist $u_0 \in \mathrm{c}(\mathrm{L}_\phi)$ and $\lambda' > 0$ such that $\rho_\phi(\lambda'(u_k^n - u_0)) \to 0$ as $k \to \infty$. Let $\lambda_0 = \min\{\lambda, \lambda'\}$. Then

$$\rho_\phi\left(\frac{\lambda}{2}(u_k - u_0)\right) \leq \rho_\phi\big(\lambda(u_k - u_k^n)\big) + \rho_\phi\big(\lambda'(u_k^n - u_0)\big) \to 0$$

as $k \to \infty$, which mean $(u_k) \in \mathrm{c}(\mathrm{L}_\phi)$. ∎

The completeness of $\mathrm{l}_\infty(\mathrm{L}_\phi)$ and Theorem 5 implies the following corollary.

Corollary. Each of $\mathrm{c}_0(\mathrm{L}_\phi)$ and $\mathrm{c}(\mathrm{L}_\phi)$ is a complete space with respect to the norm $\|.\|_{\rho_\infty}$.

The sequence space $W \subset \omega(\mathrm{L}_\phi)$ is called *K*-space if for each k, the mapping $P_k: \mathrm{W} \to \mathrm{L}_\phi$ is continuous. Furthermore, if W is complete then it is called *FK*-space.

Theorem 7. For any Orlicz function ϕ, $\mathrm{l}_\infty(\mathrm{L}_\phi)$ is an FK-space.

Proof. Given any integer k and $(u_k) \in \mathrm{l}_\infty(\mathrm{L}_\phi)$. Since $\rho_\phi(u_k) \leq \rho_\infty((u_k))$, by definition of Luxemburg norm, then we have $\|u_k\|_\phi \leq \|(u_k)\|_{\rho_\infty}$. Hence, the mapping $P_k: \mathrm{l}_\infty(\mathrm{L}_\phi) \to \mathrm{L}_\phi$, with $P_k\left((u_j)\right) = u_k$ is continuous. ∎

4. CONCLUSION

The sequence spaces we have constructed above is a generalization of the relating sequence spaces on the Banach space $\mathrm{L_p}$, for $p < 1 < \infty$. In addition, some new properties on the modular convergence are founded.

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