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Constructing Indicators of Mathematical Proof Reasoning for the Topic of Sytems of Linear Equations in Two Variables Based on Hanna-Raman Theory

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Abstract
This study aims to build a framework of indicators for mathematical proof reasoning grounded in Hanna-Raman theory, applied to the topic of Systems of Linear Equations in Two Variables (SLETV) at the senior high school level. Hanna-Raman theory draws a distinction between proof that proves, which functions to verify the truth of a statement, and proof that explains, which reveals why the statement holds. Using a Content Analysis approach combined with an Analytical-Conceptual Design, this article works through the epistemological structure of SLETV, brings together the defining features of proof within school algebra, and proposes three levels of indicators that together form a continuum running from procedural toward structural thinking. The analysis shows that Level 1 (Procedural Verification) corresponds to proof that proves, whereas Level 2 (Conceptual Justification) and Level 3 (Structural Generalization) correspond to proof that explains. The construct validity of the framework is examined through three sample problems, each accompanied by a deconstruction analysis, a mapping of relevant indicators, a comparison between proves-type and explains-type responses, and an account of likely misconceptions. The article argues that SLETV assessment, which has so far tended to focus on procedural end results, needs to shift toward valuing the process by which students construct mathematical arguments. The framework is proposed as a diagnostic tool for qualitative studies aiming to identify the proof-reasoning profiles of senior high school students.

Keywords:
Mathematical Proof; Hanna-Raman Theory; Systems of Linear Equations in Two Variables (SLETV); Deductive Reasoning; Proof that Proves; Proof that Explains.

A. INTRODUCTION

Deductive reasoning lies at the heart of what counts as genuine mathematical activity (Stylianides, 2007). Within secondary mathematics education, the ability to reason through proof is not simply a curricular requirement; it is an epistemological foundation that shapes how well students come to understand the underlying structure of mathematics itself. The Merdeka Curriculum, together with international curriculum documents such as those published by NCTM (2000), states explicitly that proof and reasoning should be an integral part of mathematics instruction at every level, senior high school included.

Classroom reality, however, tells a rather different story, one marked by a considerable gap between these curricular aspirations and the proving practices that actually take place. De Villiers (1990) and Hanna (2000) caution that proof in school

should not be reduced to a purely mechanical, procedural exercise, a condition that Hanna and Raman (2004) refer to as proof that proves, but should instead be capable of illuminating understanding, what they term proof that explains. This distinction becomes especially important once we consider how senior high school students actually construct mathematical arguments: are they simply confirming that a statement is true, or do they genuinely understand why it is true?

This issue becomes particularly visible in the topic of Systems of Linear Equations in Two Variables (SLETV), one of the cornerstones of school algebra, first introduced in grade VIII of junior secondary school and revisited in greater depth at the senior level. In practice, the teaching of SLETV tends to be narrowed quite severely: the topic is often reduced to little more than a speed drill built around a memorized elimination-substitution routine (Kieran, 2007). Students are trained to manipulate symbols algorithmically without ever being prompted to ask why the procedure works, when a system will actually have a solution, or what the solution means geometrically.

The consequences of this narrowing are considerable. When students encounter non-standard cases, an inconsistent system with no solution, say, or a dependent system with infinitely many solutions, their reasoning tends to stall (Dreyfus & Hadas, 1996). They are unable to explain why the elimination procedure produces a contradiction such as $0 = 5$, because they were never given the opportunity to develop a structural understanding of what actually happens when two linear equations are combined.

A review of the literature further reveals that Hanna-Raman-themed research is quite abundant in geometry, particularly Euclidean and dynamic geometry (Mariotti, 2006; Boero, 2007), yet remains scarce in the domain of school algebra. This scarcity is not without cause: there exists a mistaken and rather widespread assumption that school algebra, SLETV included, has little need for formal proof, since the topic is regarded as purely procedural and computational. This assumption runs counter to the findings of Tall (2002) and Sfard (1991), both of whom show that a structural understanding of algebra actually depends on a deep process of conceptual proof.

This gap is not only theoretical but also epistemological. Within the Islamic higher education context in which this study is situated, the pursuit of certain, reasoned knowledge (burhān) has long been treated as a demonstrative method in its own right, one that moves from accepted premises to necessary conclusions through valid logical structure (Arjmand, 2018). Recent scholarship on Islamic epistemology similarly argues that reasoning and logic should be cultivated as a deliberate bridge between rational and revealed sources of knowledge in contemporary Islamic education, rather than treated as a secular add-on borrowed wholesale from Western curricula (Nasir et al., 2025). Read against this backdrop, the near-absence of Hanna-Raman-based proof research in school algebra, and the almost complete absence of any explicit connection between mathematical proof and the Islamic intellectual tradition of demonstrative reasoning, together constitute the specific gap this article addresses: a framework that is theoretically grounded in mathematics education research while remaining legible to, and consistent with, the epistemological commitments of Islamic higher education institutions such as those convened under ICIED.

Building on this background, the present article sets out with a specific aim: to construct a framework of indicators capable of mapping students' patterns of proof reasoning within the SLETV domain. The framework is built upon Hanna-Raman theory as its principal analytical lens, in the hope that it can both help fill a gap in the existing literature and offer a practical contribution toward developing a more comprehensive and meaningful alternative to current SLETV assessment practice in senior high school algebra.

B. METHODS

This article was written using a Content Analysis approach combined with an Analytical-Conceptual Design. This approach was chosen because the article's main purpose is to construct a theoretical framework through the synthesis of existing literature, rather than to collect empirical data from the field. Krippendorff (2018) defines content analysis as a research technique built on a set of replicable procedures used to draw valid inferences from text, or other content, in relation to the context of its use. The work behind this article proceeded through three systematic stages, as follows.

Deconstruction Stage

At this stage, the mathematical structure of SLETV was systematically broken down from the epistemological standpoint of proof. The primary sources examined included: (1) senior high school mathematics textbooks used in Indonesia, including those aligned with the Merdeka Curriculum, (2) official curriculum documents (Learning Outcomes, Learning Objective Flows), (3) scholarly articles addressing students' conceptual errors within SLETV, and (4) literature concerning Hanna-Raman theory and school-based mathematical proof. The deconstruction focused on identifying the critical junctures at which proof, whether of the proves or the explains variety, might be drawn out within the SLETV topic.

Synthesis Stage

At this stage, the findings from the deconstruction stage were brought together with Hanna–Raman theory and the wider literature on proof reasoning (Balacheff, 1988; Dreyfus, 1999; Stylianides, 2007). This synthesis process made use of theoretical sampling (Strauss & Corbin, 1998): concepts drawn from various sources were selected on the basis of their relevance to the goal of constructing the framework. By the close of this stage, an initial formulation of the three-level continuum of indicators, together with its evidential characteristics, had taken shape.

Formulation Stage

At this stage, the initial formulation produced during synthesis was operationalized into indicators that can be observed and assessed within an actual learning context. Each indicator was worded using an operational verb reflecting the intended cognitive level (Anderson & Krathwohl, 2001). The conceptual coherence and illustrative applicability of the framework were then examined by analyzing three SLETV problems specifically designed to elicit responses across all three indicator levels. This analysis covered the deconstruction of each problem, the mapping of relevant indicators, an examination of the difference between proof that proves and proof that explains responses, and the identification of likely misconceptions.

It should be stated explicitly that, consistent with the Content Analysis and Analytical-Conceptual Design adopted here, this examination is theoretical rather than empirical: it does not involve the administration of instruments, classroom observation, or the computation of statistical reliability or validity coefficients (e.g., Cronbach's alpha, inter-rater agreement, or item discrimination indices), since no student response data were collected in the course of this study. What is offered instead is an argument for the framework's internal coherence, demonstrated through its alignment with established theory and its capacity to differentiate qualitatively distinct levels of response when applied to illustrative problems. Empirical validation of the framework, through piloting with actual student work and formal reliability testing, is proposed as a necessary next step and is discussed further in the Conclusion.

C. RESULT & DISCUSSION

Framework of Proof-Reasoning Indicators for SLETV

Working through the stages outlined above produced a three-level framework of indicators, presented in Table 1 below.

Table 1. Framework of Proof-Reasoning Indicators for SLETV Based on Hanna-Raman Theory

Level	Level Name	Indicator Descriptor	Characteristics of Student Response	Hanna-Raman Category
1	**Procedural Verification**	1a. The student can correctly carry out an elimination or substitution procedure to find the value of a variable. 1b. The student can check a solution's correctness by substituting the variable values back into the original equations. 1c. The student can identify which method was used (elimination, substitution, or graphing) even without stating its theoretical basis.	A proof is taken as valid once the procedural steps are syntactically correct, even if the student cannot explain the relationship between the coefficients (e.g., why parallel lines have no solution).	**PROOF THAT PROVES**
2	**Conceptual Justification**	2a. The student can explain why an algebraic operation (such as multiplying by a constant) leaves the solution set unchanged. 2b. The student can connect the steps of the algebraic procedure to their geometric representation on a graph. 2c. The student can build an argument about the consistency of a system's solution based on the relationship among its coefficients.	The student supplies a conceptual reason for every critical step. Responses show a connection between symbolic and geometric representations.	**PROOF THAT EXPLAINS**
3	**Structural Generalization**	3a. The student can prove, in general terms, the conditions	Arguments are built through strict deductive reasoning, and the student	**PROOF THAT EXPLAINS**

Level	Level Name	Indicator Descriptor	Characteristics of Student Response	Hanna-Raman Category
		characterizing an SLETV's solution (unique, infinite, or none) using the concept of determinant or linear independence. 3b. The student can construct a deductive proof of the elimination method's validity based on the basic axioms of the real number system. 3c. The student can apply a mastered proof scheme to a new, analogous context (such as a system of equations with a parameter).	recognizes the importance of formal definitions and the necessary/sufficient conditions within a linear system.	**(Advanced Level)**

Source: adapted from Hanna & Raman (2004), Balacheff (1988), and Stylianides (2007)

Narrative Analysis of the Framework

The framework presented in Table 1 carries a number of epistemological and pedagogical strengths that deserve to be spelled out explicitly. First, the framework is inclusive and hierarchical in nature: each higher level does not replace the level below it but rather absorbs it. A student operating at Level 3 has, by implication, already mastered Levels 1 and 2. This sets the framework apart from deficit models, which do little more than identify what students are lacking.

Second, the framework maps quite precisely onto the Hanna-Raman categories. Level 1 logically corresponds to proof that proves, since the procedural verification occurring at this level, although it produces a correct outcome, gives no insight into why the system has that particular solution. Level 2, on the other hand, corresponds to proof that explains, since at this level students begin to construct arguments that expose causal relationships, why the elimination operation is legitimate, why a system with parallel gradients has no solution. Level 3 deepens this explanatory character further still by generalizing the argument into a form that is universal and deductively formal.

Third, the framework is responsive to the cultural context of learning in Indonesia. The Merdeka Curriculum places considerable weight on the Pancasila Student Profile, particularly its dimensions of critical and creative reasoning. Levels 2 and 3 of this framework put those dimensions directly into practice within mathematics learning, so that the framework carries relevance not only to international theory but also to national curricular priorities.

Problem Analysis and Framework Validation

Problem 1: A Consistent System with a Unique Solution

Problem 1:
Given the following system of linear equations:
$2x + 3y = 12$ ... (1)
$4x - y = 2$ ... (2)
a) Determine the solution set of the system.
b) Explain why multiplying equation (2) by 3 and then adding it to equation (1) is a valid operation that does not change the solution set.
c) Prove, in general, that if (x_0, y_0) is a solution of the system above, then (x_0, y_0) is also a solution of every linear combination of the two equations.

Deconstruction Analysis: Problem 1 was designed in tiers so as to draw out all three indicator levels at once. Part (a) is a standard procedural task aimed at Level 1. Part (b) is a conceptual-justification task aimed at Level 2, requiring students to articulate the equality property that underlies whether the operation is legitimate. Part (c) is a structural-generalization task aimed at Level 3, requiring students to prove that a system's solution remains unchanged under a linear combination, which forms the very foundation of the Gaussian elimination method.

Indicator Alignment:

- Part (a) → Indicators 1a, 1b.
- Part (b) → Indicator 2a.
- Part (c) → Indicator 3b.

Analysis of Proves vs. Explains Responses: A student operating at the proof that proves level will complete part (a) correctly but will stall, or offer a tautological response, on part (b): "because you're allowed to," or "because that's how elimination works." Such students have no access to the underlying principle. A student at the proof that explains level, by contrast, will state explicitly that if $A = B$ and $C = D$, then $A + kC = B + kD$ for any constant k, and that it is precisely this principle that guarantees the validity of the elimination operation.

Likely Misconceptions: A very common misconception on this type of problem is the belief that multiplying one equation must be "balanced" by a similar multiplication applied to the other equation. This misconception reflects a mistaken view of the two equations as independent, standalone entities, rather than as a system bound together by a shared solution set.

Problem 2: An Inconsistent System

Problem 2:
A student attempts to solve the following system using the elimination method:
$3x + 6y = 12$ ... (1)
$x + 2y = 5$ ... (2)
After performing elimination, the student arrives at $0 = -3$. Confused, the student concludes, "I must have made an arithmetic mistake."
a) Verify whether the student's elimination steps are correct.
b) Explain conceptually why this system has no solution, using both a geometric and an algebraic representation.
c) Prove that for any system $ax + by = c$ and $kax + kby = d$ where $d \neq kc$, the system must be inconsistent.

Deconstruction Analysis: Problem 2 is designed to expose the case in which procedural reduction proves most damaging, an inconsistent system. Part (a) targets Level 1 by verifying the correctness of the elimination procedure. Part (b) targets Level 2, requiring a conceptual explanation that draws on two distinct representations,

geometric (parallel lines) and algebraic (contradiction). Part (c) targets Level 3, requiring a general proof built on universal quantification.

Indicator Alignment:

- Part (a) → Indicator 1b.
- Part (b) → Indicators 2b, 2c.
- Part (c) → Indicator 3a.

Analysis of Proves vs. Explains Responses: A proof that proves student will correctly identify that the elimination steps carried out in the problem are valid (part a), yet will be unable to explain why the result $0 = -3$ means the system has no solution. They might say "because the result is wrong" or "because it doesn't make sense." A proof that explains student will articulate that $0 = -3$ is a logical contradiction that can never hold true, meaning no values of x and y can satisfy both equations simultaneously. Geometrically, they will explain that the two equations represent parallel lines, equal gradient, different intercepts, that never intersect.

Likely Misconceptions: The most common misconception here is the assumption that a contradictory result appearing during elimination always signals an arithmetic error. This misconception follows directly from a procedural orientation: students are never equipped with the understanding that elimination can expose an inconsistency intrinsic to the system itself, rather than simply processing numbers toward a solution.

Problem 3: A Dependent System (Infinitely Many Solutions)

Problem 3:
Consider the following system:
$2x - 4y = 8$ ... (1)
$-x + 2y = -4$ ... (2)
a) Use the elimination method and explain what happens when the two equations are eliminated.
b) State the solution set of this system in parametric form and give a conceptual justification for why the system has infinitely many solutions.
c) Prove that if one equation in a system is a scalar multiple of the other, the system must either have infinitely many solutions or no solution at all. State the condition that distinguishes the two cases.

Deconstruction Analysis: Problem 3 tests students' understanding of the case least often addressed in conventional SLETV instruction, a dependent system. Part (a) targets Level 1 through procedural execution and observation of the result $0 = 0$. Part (b) targets Level 2, requiring a parametric representation together with conceptual justification. Part (c) targets Level 3, requiring a general proof that distinguishes a dependent system from an inconsistent one, a proof that depends on an understanding of linear independence.

Indicator Alignment:

- Part (a) → Indicators 1a, 1c.
- Part (b) → Indicators 2b, 2c.
- Part (c) → Indicators 3a, 3c.

Analysis of Proves vs. Explains Responses: With this type of problem, the gap between proof that proves and proof that explains becomes especially pronounced. A student at the procedural level will be entirely unable to interpret the result $0 = 0$, or will misinterpret it as meaning "the system has no solution" (echoing the misconception seen in Problem 2). A student at the explanatory level will be able to explain that $0 = 0$ is an identity that always holds true, meaning that equation (2) contributes no new information beyond what equation (1) already provides, so that infinitely many solutions follows as a logical consequence.

Likely Misconceptions: A critical misconception that appears frequently is reading $0 = 0$ as evidence that "every value of x and y is a solution," without restricting the solution to the line defined by either equation. This misconception points to a poor grasp of the constrained solution set, the idea that infinitely many solutions does not mean unlimited solutions across the entire coordinate plane, but rather solutions unlimited along one particular line.

Discussion

Taken together, the framework and the three illustrative problems point to a single central argument: SLETV assessment that stops at a correct numerical answer captures only a fraction of what students actually know. Consistent with Hanna and Raman's (2004) original distinction, the procedural verification typical of Level 1 is logically sufficient but epistemically thin, it confirms that a solution is correct without ever asking a student to account for why the system behaves as it does. The shift toward Level 2 and Level 3 responses is precisely the shift Stylianides (2007) has in mind when he argues that proof in school mathematics should be an integral part of doing mathematics rather than an advanced add-on reserved for geometry.

This distinction resonates strongly with the Islamic epistemological tradition in which this study is situated. Classical Islamic logic distinguishes knowledge obtained through burhān, demonstrative proof that yields certainty (yaqīn) by tracing a conclusion back to self-evident or already-established premises, from knowledge obtained through mere assertion or authority (Arjmand, 2018). A student who produces only a proof that proves is, in this sense, working with an unexamined premise: the answer is accepted because the procedure yields it, not because its necessity has been grasped. A student who produces a proof that explains, by contrast, is engaged in something much closer to the spirit of burhān, tracing why a conclusion must follow from what is already known. Nasir et al. (2025) similarly argue that logic and reasoning in Islamic education should function as a deliberate discipline of the intellect rather than a purely instrumental problem-solving skill; the SLETV framework proposed here can be read as one concrete, subject-specific attempt to operationalize that discipline within school mathematics.

The hierarchical, inclusive structure of the framework also resonates with Sfard's (1991) account of reification: a student who has reached Level 3 has not abandoned procedural fluency but has folded it into a structural view of the equation system as an object with its own properties. This reading is consistent with Kieran's (2007) observation that the transition from arithmetic to algebra stalls precisely when students are never pushed past the operational stage. The misconceptions identified across the three problems also line up with Balacheff's (1988) description of naive empiricism, and, in Weber and Alcock's (2004) terms, with syntactic rather than semantic proof production: students manipulate symbols correctly without the conceptual grounding that would let them reason about what the symbols represent.

Placed alongside recent Indonesian research on mathematical reasoning, this pattern is not unique to SLETV. Agusantia and Juandi's (2022) systematic review of analogical reasoning research in Indonesian mathematics classrooms likewise finds that reasoning skills are seldom assessed as a distinct construct, with instruction and evaluation defaulting instead to procedural correctness. The present framework contributes to this broader conversation by offering a subject-specific, theory-grounded alternative for one topic, SLETV, that can plausibly be adapted to other algebra topics facing the same assessment gap.

D. CONCLUSION

This article has succeeded in constructing a framework of mathematical proof-reasoning indicators grounded in Hanna-Raman theory, designed specifically for the SLETV topic at the senior high school level. The framework comprises three hierarchical levels, Procedural Verification, Conceptual Justification, and Structural Generalization, which together form a continuum running from procedural toward structural thinking.

The central argument advanced in this article is that this Hanna-Raman-based indicator framework succeeds in shifting SLETV assessment away from its original emphasis on final outcomes, a correct numerical answer, toward an emphasis on the process of constructing mathematical arguments. This shift is not simply a technical adjustment to question format or scoring rubrics; it represents a fundamental epistemological shift in what mathematics learning at the secondary level ought to be aiming for.

Through the analysis of three specially designed problems, this article has also shown that SLETV, long mistakenly regarded as a topic with little need for formal proof, in fact holds considerable potential for proof-based learning. The cases of inconsistent and dependent systems, in particular, serve as a remarkably transparent window into whether a student is operating at the level of proof that proves or proof that explains, since only students with structural understanding are able to interpret and meaningfully explain both cases.

For classroom practice, this framework can be used as a diagnostic rubric rather than a conventional answer key. Concretely, teachers can: (1) replace single-answer SLETV quizzes with tiered items of the kind illustrated in Problems 1 to 3, each containing a procedural part, a conceptual-justification part, and a general-proof part; (2) score student responses against the three indicator levels in Table 1 rather than against correctness alone, so that a numerically correct but unexplained answer is recognized as Level 1 rather than as full mastery; (3) use the misconception patterns identified for inconsistent and dependent systems as a basis for targeted remediation, since these two cases are where procedural-only instruction most visibly breaks down; and (4) treat students' movement from proof that proves toward proof that explains as an explicit learning goal in lesson planning and professional development, in a manner consistent with the emphasis on reasoned, demonstrative understanding found in the Islamic intellectual tradition of burhān.

This study is nonetheless conceptual and analytical in design, and its limitations should be stated plainly. The three-level framework and the accompanying problem analyses were developed through content analysis and theoretical synthesis rather than through the collection of field data; no instruments were administered to students, and no statistical reliability or validity coefficients were computed. The claims made here concern the framework's conceptual coherence and its capacity, on paper, to differentiate qualitatively distinct kinds of student reasoning, not its empirically demonstrated reliability in actual classrooms. A natural and necessary next step is therefore an empirical follow-up study, ideally a mixed-methods design, that pilots the tiered problems with senior high school students, gathers their written and verbal reasoning, and examines the framework's inter-rater reliability and classification consistency before it is recommended for wider assessment use.

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